

Quantum Hashing via Classical ϵ -universal Hashing Constructions

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Abstract

We define the concept of a quantum hash generator and offer a design, which allows one to build a large number of different quantum hash functions. The construction is based on composition of a classical ϵ -universal hash family and a given family of functions – quantum hash generators.

The relationship between ϵ -universal hash families and error-correcting codes give possibilities to build a large amount of different quantum hash functions. In particular, we present quantum hash function based on Reed-Solomon code, and we proved, that this construction is optimal in the sense of number of qubits needed.

Using the relationship between ϵ -universal hash families and Freivalds' fingerprinting schemas we present explicit quantum hash function and prove that this construction is optimal with respect to the number of qubits needed for the construction.

Keywords: quantum hashing, quantum hash function, ϵ -universal hashing, error-correcting codes.

1 Introduction

Quantum computing is inherently a very mathematical subject, and the discussions of how quantum computers can be more efficient than classical computers in breaking encryption algorithms started since Shor invented his famous quantum algorithm. The answer of the cryptography community is “Post-quantum cryptography”, which refers to research on problems (usually public-key cryptosystems) that are no more efficiently breakable using quantum computers than by classical computer architectures. Currently post-quantum cryptography includes several approaches, in particular, hash-based signature schemes such as Lamport signatures and Merkle signature schemes.

Hashing itself is an important basic concept for the organization transformation and reliable transmission of information. The concept known as “universal hashing” was invented by Carter and Wegman [7] in 1979. In 1994 a relationship was discovered between

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ϵ -universal hash families and error-correcting codes [5]. In [16] Wigderson characterizes universal hashing as being a tool which “should belong to the fundamental bag of tricks of every computer scientist”.

Gottesman and Chuang proposed a quantum digital system [9], based on quantum mechanics. Their results are based on quantum a fingerprinting technique and add “quantum direction” for post-quantum cryptography. Quantum fingerprints have been introduced by Buhrman, Cleve, Watrous and de Wolf in [6]. Gavinsky and Ito [8] viewed quantum fingerprints as cryptographic primitives.

In [2, 3] we considered quantum fingerprinting as a construction for binary hash functions and introduced a non-binary hash function. The quantum hashing proposed a suitable one-way function for quantum digital signature protocol from [9]. For more introductory information we refer to [2].

In this paper, we define the concept of a quantum hash generator and offer a design, which allows one to build different quantum hash functions. The construction is based on the composition of classical ϵ -universal hash family with a given family of functions – quantum hash generator.

The construction proposed combines the properties of robust presentation of information by classical error-correcting codes together with the possibility of highly compressed presentation of information by quantum systems.

The relationship between ϵ -universal hash families and error-correcting codes give possibilities to build a large amount of different quantum hash functions. In particular, we present quantum hash function based on Reed-Solomon code, and we proved, that this construction is optimal in the sense of number of qubits needed.

Using the relationship between ϵ -universal hash families and Freivalds’ fingerprinting schemas we present an explicit quantum hash function and prove that this construction is optimal with respect to of number of qubits needed for the construction.

1.1 Definitions and Notations

We begin by recalling some definitions of classical hash families from [13]. Given a domain \mathbb{X} , $|\mathbb{X}| = K$, and a range \mathbb{Y} , $|\mathbb{Y}| = M$, (typically with $K \geq M$), a hash function f is a map

$$f : \mathbb{X} \rightarrow \mathbb{Y},$$

that hash *long* inputs to *short* outputs.

We let q to be a prime power and \mathbb{F}_q be a finite field of order q . Let Σ^k be a set of words of length k over a finite alphabet Σ . In the paper we let $\mathbb{X} = \Sigma^k$, or $\mathbb{X} = \mathbb{F}_q$, or $\mathbb{X} = (\mathbb{F}_q)^k$, and $\mathbb{Y} = \mathbb{F}_q$. A hash family is a set $F = \{f_1, \dots, f_N\}$ of hash functions $f_i : \mathbb{X} \rightarrow \mathbb{Y}$.

ϵ universal hash family. A hash family F is called an ϵ -universal hash family if for any two distinct elements $w, w' \in \mathbb{X}$, there exist at most ϵN functions $f \in F$ such that $f(w) = f(w')$. We will use the notation ϵ -U ($N; K, M$) as an abbreviation for ϵ -universal hash family.

Clearly we have, that if function the f is chosen uniformly at random from a given ϵ -U $(N; K, M)$ hash family F , then the probability that any two distinct words collide under f is at most ϵ .

The case of $\epsilon = 1/N$ is known as universal hashing.

Classical-quantum function. The notion of a quantum function was considered in [11]. In this paper we use the following variant of a quantum function. First recall that mathematically a qubit $|\psi\rangle$ is described as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers, satisfying $|\alpha|^2 + |\beta|^2 = 1$. So, a qubit may be presented as a unit vector in the two-dimensional Hilbert complex space \mathcal{H}^2 . Let $s \geq 1$. Let $(\mathcal{H}^2)^{\otimes s}$ be the 2^s -dimensional Hilbert space, describing the states of s qubits, i.e. $(\mathcal{H}^2)^{\otimes s}$ is made up of s copies of a single qubit space \mathcal{H}^2

$$(\mathcal{H}^2)^{\otimes s} = \mathcal{H}^2 \otimes \dots \otimes \mathcal{H}^2 = \mathcal{H}^{2^s}.$$

For $K = |\mathbb{X}|$ and integer $s \geq 1$ we define a $(K; s)$ classical-quantum function to be a map of the elements $w \in \mathbb{X}$ to quantum states $|\psi(w)\rangle \in (\mathcal{H}^2)^{\otimes s}$

$$\psi : \mathbb{X} \rightarrow (\mathcal{H}^2)^{\otimes s}. \quad (1)$$

We will also use the notation $\psi : w \mapsto |\psi(w)\rangle$ for ψ .

2 Quantum hashing

What we need to define for quantum hashing and what is implicitly assumed in various papers (see for example [2] for more information) is a collision resistance property. However, there is still no such notion as *quantum collision*. The reason why we need to define it is the observation that in quantum hashing there might be no collisions in the classical sense: since quantum hashes are quantum states they can store an arbitrary amount of data and can be different for different messages. But the procedure of comparing those quantum states implies measurement, which can lead to collision-type errors.

So, a *quantum collision* is a situation when a procedure that tests the equality of quantum hashes and outputs “true”, while hashes are different. This procedure can be a well-known SWAP-test (see for example [2] for more information and citations) or something that is adapted for specific hashing function. Anyway, it deals with the notion of distinguishability of quantum states. Since non-orthogonal quantum states cannot be perfectly distinguished, we require them to be “nearly orthogonal”.

- For $\delta \in (0, 1/2)$ we call a function

$$\psi : \mathbb{X} \rightarrow (\mathcal{H}^2)^{\otimes s}$$

δ -resistant, if for any pair w, w' of different elements,

$$|\langle \psi(w) | \psi(w') \rangle| \leq \delta.$$

Theorem 1 *Let $\psi : \mathbb{X} \rightarrow (\mathcal{H}^2)^{\otimes s}$ be a δ -resistant function. Then*

$$s \geq \log \log |\mathbb{X}| - \log \log \left(1 + \sqrt{2/(1-\delta)} \right) - 1.$$

Proof. First we observe, that from the definition $|||\psi\rangle|| = \sqrt{\langle\psi|\psi\rangle}$ of the norm it follows that

$$|||\psi\rangle - |\psi'\rangle||^2 = |||\psi\rangle||^2 + |||\psi'\rangle||^2 - 2\langle\psi|\psi'\rangle.$$

Hence for an arbitrary pair w, w' of different elements from \mathbb{X} we have that

$$|||\psi(w)\rangle - |\psi(w')\rangle|| \geq \sqrt{2(1-\delta)}.$$

We let $\Delta = \sqrt{2(1-\delta)}$. For short we let $(\mathcal{H}^2)^{\otimes s} = V$ in this proof. Consider a set $\Phi = \{|\psi(w)\rangle : w \in \mathbb{X}\}$. If we draw spheres of radius $\Delta/2$ with centres $|\psi\rangle \in \Phi$ then spheres do not pairwise intersect. All these K spheres are in a large sphere of radius $1 + \Delta/2$. The volume of a sphere of radius r in V is $cr^{2^{s+1}}$ for the complex space V . The constant c depends on the metric of V . From this we have, that the number K is bonded by the number of “small spheres” in the “large sphere”

$$K \leq \frac{c(1 + \Delta/2)^{2^{s+1}}}{c(\Delta/2)^{2^{s+1}}}.$$

Hence

$$s \geq \log \log K - \log \log \left(1 + \sqrt{2/(1-\delta)} \right) - 1.$$

□

The notion of δ -resistance naturally leads to the following notion of quantum hash function.

Definition 1 (Quantum hash function) *Let K, s be positive integers and $K = |\mathbb{X}|$. We call a map*

$$\psi : \mathbb{X} \rightarrow (\mathcal{H}^2)^{\otimes s}$$

an δ -resistant $(K; s)$ quantum hash function if ψ is a δ -resistant function.

We use the notation $\delta\text{-R}(K; s)$ as an abbreviation for δ -resistant $(K; s)$ quantum hash functions.

3 Generator for Quantum Hash Functions

In this section we present two constructions of quantum hash functions and define notion of quantum hash function generator, which generalizes these constructions.

3.1 Binary quantum hashing.

One of the first explicit quantum hash functions was defined in [6]. Originally the authors invented a construction called “quantum fingerprinting” for testing the equality of two words for a quantum communication model. The cryptography aspects of quantum fingerprinting are presented in [8]. The quantum fingerprinting technique is based on binary error-correcting codes. Later this construction was adopted for cryptographic purposes. Here we present the quantum fingerprinting construction from the quantum hashing point of view.

An (n, k, d) error-correcting code is a map

$$C : \Sigma^k \rightarrow \Sigma^n$$

such that, for any two distinct words $w, w' \in \Sigma^k$, the Hamming distance between code words $C(w)$ and $C(w')$ is at least d . The code is binary if $\Sigma = \{0, 1\}$.

The construction of a quantum hash function based on quantum fingerprinting is as follows.

- Let $c > 1$ and $\delta < 1$. Let k be a positive integer and $n > k$. Let $E : \{0, 1\}^k \rightarrow \{0, 1\}^n$ be an (n, k, d) binary error-correcting code with Hamming distance $d \geq (1 - \delta)n$.
- Define a family of functions $F_E = \{E_1, \dots, E_n\}$, where $E_i : \{0, 1\}^k \rightarrow \mathbb{F}_2$ is defined by the rule: $E_i(w)$ is the i -th bit of the code word $E(w)$.
- Let $s = \log n + 1$. Define the classical-quantum function $\psi_{F_E} : \{0, 1\}^k \rightarrow (\mathcal{H}^2)^{\otimes s}$, determined by a word w as

$$\psi_{F_E}(w) = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i\rangle |E_i(w)\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i\rangle \left(\cos \frac{\pi E_i(w)}{2} |0\rangle + \sin \frac{\pi E_i(w)}{2} |1\rangle \right),$$

For $s = \log n + 1$, the function ψ_{F_E} is an δ -R $(2^k; s)$ quantum hash function, that is, for two different words w, w' we have

$$|\langle \psi_{F_E}(w) | \psi_{F_E}(w') \rangle| \leq \delta n/n = \delta.$$

Observe, that the authors in [6] propose, for the first choice of such binary codes, Justesen codes with $n = ck$, which give $\delta < 9/10 + 1/(15c)$ for any chosen $c > 2$. Next we observe, that the above construction of a quantum hash function needs $\log n + 1$ qubits for the fixed $\delta \approx 9/10 + 1/(15c)$. This number of qubits is good enough in the sense of the lower bound of Theorem 1.

A non-binary quantum hash function is presented in [2] and is based on the construction from [1].

3.2 Non-binary quantum hashing.

We present the non-binary quantum hash function from [2] in the following form. For a field \mathbb{F}_q , let $B = \{b_1, \dots, b_T\} \subseteq \mathbb{F}_q$. For every $b_j \in B$ and $w \in \mathbb{F}_q$, define a function $h_j : \mathbb{F}_q \rightarrow \mathbb{F}_q$ by the rule

$$h_j(w) = b_j w \pmod{q}.$$

Let $H = \{h_1, \dots, h_T\}$ and $t = \log T$. We define the classical-quantum function

$$\psi_H : \mathbb{F}_q \rightarrow (\mathcal{H}^2)^{\otimes(t+1)}$$

by the rule

$$|\psi_H(w)\rangle = \frac{1}{\sqrt{T}} \sum_{j=1}^T |j\rangle \left(\cos \frac{2\pi h_j(w)}{q} |0\rangle + \sin \frac{2\pi h_j(w)}{q} |1\rangle \right).$$

The following is proved in [2].

Theorem 2 *Let q be a prime power and \mathbb{F}_q be a field. Then, for arbitrary $\delta > 0$, there exists a set $B = \{b_1, \dots, b_T\} \subseteq \mathbb{F}_q$ (and, therefore, a corresponding family $H = \{h_1, \dots, h_T\}$ of functions) with $T = \lceil (2/\delta^2) \ln(2q) \rceil$, such that the quantum function ψ_H is a δ - $R(q; t+1)$ quantum hash function.*

In the rest of the paper we use the notation $H_{\delta,q}$ to denote this family of functions from Theorem 2 and the notation $\psi_{H_{\delta,q}}$ to denote the corresponding quantum function.

Observe, that the above construction of the quantum hash function $\psi_{H_{\delta,q}}$ needs $t+1 \leq \log \log 2q + 2 \log 1/\delta + 3$ qubits. This number of qubits is good enough in the sense of the lower bound of Theorem 1.

Numerical results on $\psi_{H_{\delta,q}}$ are presented in [2].

3.3 Quantum hash generator

The above two constructions of quantum hash functions are using certain controlled rotations of target qubits. These transformations are generated by the corresponding discrete functions from a specific family of functions (F_E and $H_{\delta,q}$ respectively).

These constructions lead to the following definition.

Definition 2 (Quantum hash generator) *Let $K = |\mathbb{X}|$ and let $G = \{g_1, \dots, g_D\}$ be a family of functions $g_j : \mathbb{X} \rightarrow \mathbb{F}_q$. Let $\ell \geq 1$ be an integer. For $g \in G$ let ψ_g be a classical-quantum function $\psi_g : \mathbb{X} \rightarrow (\mathcal{H}^2)^{\otimes \ell}$ determined by the rule*

$$\psi_g : w \mapsto |\psi_g(w)\rangle = \sum_{i=1}^{2^\ell} \alpha_i(g(w)) |i\rangle, \quad (2)$$

where the amplitudes $\alpha_i(g(w))$, $i \in \{1, \dots, 2^\ell\}$, of the state $|\psi_g(w)\rangle$ are determined by $g(w)$.

Let $d = \log D$. We define a classical-quantum function $\psi_G : \mathbb{X} \rightarrow (\mathcal{H}^2)^{\otimes(d+\ell)}$ by the rule

$$\psi_G : w \mapsto |\psi_G(w)\rangle = \frac{1}{\sqrt{D}} \sum_{j=1}^D |j\rangle |\psi_{g_j}(w)\rangle. \quad (3)$$

We say that the family G generates the δ -R $(K; d + \ell)$ quantum hash function ψ_G and we call G a δ -R $(K; d + \ell)$ quantum hash generator, if ψ_G is a δ -R $(K; d + \ell)$ quantum hash function.

According to Definition 2 the family $F_E = \{E_1, \dots, E_n\}$ from Section 3.1 is a δ -R $(2^k; \log n + 1)$ quantum hash generator and the family $H_{\delta,q}$ from Section 3.2 is δ -R $(q; t + 1)$ quantum hash generator.

4 Quantum Hashing via Classical ϵ -Universal Hashing Constructions

In this section we present a construction of a quantum hash generator based on the composition of an ϵ -universal hash family with a given quantum hash generator. We begin with the definitions and notation that we use in the rest of the paper.

Let $K = |\mathbb{X}|$, $M = |\mathbb{Y}|$. Let $F = \{f_1, \dots, f_N\}$ be a family of functions, where

$$f_i : \mathbb{X} \rightarrow \mathbb{Y}.$$

Let q be a prime power and \mathbb{F}_q be a field. Let $H = \{h_1, \dots, h_T\}$ be a family of functions, where

$$h_j : \mathbb{Y} \rightarrow \mathbb{F}_q.$$

For $f \in F$ and $h \in H_B$, define composition $g = f \circ h$,

$$g : \mathbb{X} \rightarrow \mathbb{F}_q,$$

by the rule

$$g(w) = (f \circ h)(w) = h(f(w)).$$

Define composition $G = F \circ H$ of two families F and H as follows.

$$G = \{g_{ij} = f_i \circ h_j : i \in I, j \in J\},$$

where $I = \{1, \dots, N\}$, $J = \{1, \dots, T\}$.

Theorem 3 *Let $F = \{f_1, \dots, f_N\}$ be an ϵ -U $(N; K, M)$ hash family. Let $\ell \geq 1$. Let $H = \{h_1, \dots, h_T\}$ be a δ -R $(M; \log T + \ell)$ quantum hash generator. Let $\log K > \log N + \log T + \ell$.*

Then the composition $G = F \circ H$ is an Δ -R $(K; s)$ quantum hash generator, where

$$s = \log N + \log T + \ell \quad (4)$$

and

$$\Delta \leq \epsilon + \delta. \quad (5)$$

Proof. The δ -R $(M; \log T + \ell)$ quantum hash generator H generates the δ -R $(M; \log T + \ell)$ quantum hash function

$$\psi_H : v \mapsto \frac{1}{\sqrt{T}} \sum_{j \in J} |j\rangle |\psi_{h_j}(v)\rangle. \quad (6)$$

For $s = \log N + \log T + \ell$, using the family G , define the map

$$\psi_G : \mathbb{X} \rightarrow (\mathcal{H}^2)^{\otimes s}$$

by the rule

$$|\psi_G(w)\rangle = \frac{1}{\sqrt{N}} \sum_{i \in I} |i\rangle \otimes |\psi_H(f_i(w))\rangle. \quad (7)$$

We show the Δ resistance of ψ_G .

Consider a pair w, w' of different elements from \mathbb{X} and their inner product $\langle \psi_G(w) | \psi_G(w') \rangle$. Using the linearity of the inner product we have that

$$\langle \psi_G(w) | \psi_G(w') \rangle = \frac{1}{N} \sum_{i \in I} \langle \psi_H(f_i(w)) | \psi_H(f_i(w')) \rangle.$$

We define two sets of indexes I_{bad} and I_{good} :

$$I_{bad} = \{i \in I : f_i(w) = f_i(w')\}, \quad I_{good} = \{i \in I : f_i(w) \neq f_i(w')\}.$$

Then we have

$$\begin{aligned} |\langle \psi_G(w) | \psi_G(w') \rangle| &\leq \frac{1}{N} \sum_{i \in I_{bad}} |\langle \psi_H(f_i(w)) | \psi_H(f_i(w')) \rangle| \\ &\quad + \frac{1}{N} \sum_{i \in I_{good}} |\langle \psi_H(f_i(w)) | \psi_H(f_i(w')) \rangle|. \end{aligned} \quad (8)$$

The hash family F is ϵ -universal, hence

$$|I_{bad}| \leq \epsilon N.$$

The quantum function $\psi_H : \mathbb{Y} \rightarrow (\mathcal{H}^2)^{\log T + \ell}$ is δ -resistant, hence for an arbitrary pair v, v' of different elements from \mathbb{Y} one has

$$|\langle \psi_H(v) | \psi_H(v') \rangle| \leq \delta.$$

Finally from (8) and the above two inequalities we have that

$$|\langle \psi_G(w) | \psi_G(w') \rangle| \leq \epsilon + \frac{|I_{good}|}{N} \delta \leq \epsilon + \delta.$$

The last inequality proves Δ -resistance of $\psi_G(w)$ (say for $\Delta = \epsilon + \delta(|I_{good}|/N)$) and proves the inequality (5).

To finish the proof of the theorem it remains to show that the function ψ_G can be presented in the form displayed in (3). From (6) and (7) we have that

$$|\psi_G(w)\rangle = \frac{1}{\sqrt{N}} \sum_{i \in I} |i\rangle \otimes \left(\frac{1}{\sqrt{T}} \sum_{j \in J} |j\rangle |\psi_{h_j}(f_i(w))\rangle \right).$$

Using the notation from (2) the above expression can be presented in the following form (3).

$$|\psi_G(w)\rangle = \frac{1}{\sqrt{NT}} \sum_{i \in I, j \in J} |ij\rangle |\psi_{g_{ij}}(w)\rangle,$$

here $|ij\rangle$ denotes a basis quantum state, where ij is treated as a concatenation of the binary representations of i and j . \square

5 Explicit Constructions of Quantum Hash Functions Based on Classical Universal Hashing

The following statement is a corollary of Theorem 3 and a basis for explicit constructions of quantum hash functions in this section. Let q be a prime power and \mathbb{F}_q be a field. Let $\delta \in (0, 1)$. Let $H_{\delta,q}$ be the family of functions from Theorem 2. Let $|\mathbb{X}| = K$.

Theorem 4 *Let $F = \{f_1, \dots, f_N\}$ be an ϵ -U $(N; K, q)$ hash family, where $f_i : \mathbb{X} \rightarrow \mathbb{F}_q$. Then for arbitrary $\delta > 0$, family $G = F \circ H_{\delta,q}$ is a Δ -R $(K; s)$ quantum hash generator, where*

$$s \leq \log N + \log \log q + 2 \log 1/\delta + 3$$

and

$$\Delta \leq \epsilon + \delta.$$

Proof. We take the family $H_{\delta,q} = \{h_1, \dots, h_T\}$, where $h_i : \mathbb{F}_q \rightarrow \mathbb{F}_q$, $T = \lceil (2/\delta^2) \ln(2q) \rceil$, $\ell = 1$, and $s = \log T + 1 \leq \log n + \log \log q + 2 \log 1/\delta + 3$. $H_{\delta,q}$ is δ -R $(q; s)$ quantum hash generator. According to Theorem 3 the composition $G = F \circ H_{\delta,q}$ is a Δ -R $(K; s)$ quantum hash generator with the stated parameters. \square

5.1 Quantum hashing from universal linear hash family

The next hash family is folklore and was displayed in several papers and books. See the paper [14] and the book [15] for more information.

- Let k be a positive integer and let q be a prime power. Let $\mathbb{X} = (\mathbb{F}_q)^k \setminus \{(0, \dots, 0)\}$. For every vector $a \in (\mathbb{F}_q)^k$ define hash function $f_a : \mathbb{X} \rightarrow \mathbb{F}_q$ by the rule

$$f_a(w) = \sum_{i=1}^k a_i w_i.$$

Then $F_{lin} = \{f_a : a \in (\mathbb{F}_q)^k\}$ is an $(1/q)$ -U $(q^k; (q^k - 1); q)$ hash family (universal hash family).

Theorem 5 *Let k be a positive integer, let q be a prime power. Then for arbitrary $\delta \in (0, 1)$ composition $G = F_{lin} \circ H_{\delta, q}$ is a Δ -R $(q^k; s)$ quantum hash generator with $\Delta \leq (1/q) + \delta$ and $s \leq k \log q + \log \log q + 2 \log 1/\delta + 3$.*

Proof. According to Theorem 4 function ψ_G is Δ -R $(q^k; s)$ quantum hash function with the parameters stated in the theorem. \square

Remark 1 *Note, that from Theorem 1 we have that*

$$s \geq \log \log |\mathbb{X}| + \log \log \left(1 + \sqrt{2/(1 - \delta)}\right) - 1 \geq \log k + \log \log q - \log \log \left(1 + \sqrt{2/(1 - \delta)}\right) - 1.$$

This lower bound shows that the quantum hash function ψ_G is not asymptotically optimal in the sense of number of qubits used for the construction.

5.2 Quantum hashing based on Freivalds' fingerprinting

For a fixed positive constant k let $\mathbb{X} = \{0, 1\}^k$. Let $c > 1$ be a positive integer and let $M = ck \ln k$. Let $\mathbb{Y} = \{0, 1, \dots, M - 1\}$.

For the i -th prime $p_i \in \mathbb{Y}$ define a function (fingerprint)

$$f_i : \mathbb{X} \rightarrow \mathbb{Y}$$

by the rule

$$f_i(w) = w \pmod{p_i}.$$

Here we treat a word $w = w_0 w_1 \dots w_{k-1}$ also as an integer $w = w_0 + w_1 2 + \dots + w_{k-1} 2^{k-1}$. Consider the set

$$F_M = \{f_1, \dots, f_{\pi(M)}\}$$

of fingerprints. Here $\pi(M)$ denotes the number of primes less than or equal to M . Note that then $\pi(M) \sim M/\ln M$ as $M \rightarrow \infty$. Moreover,

$$\frac{M}{\ln M} \leq \pi(M) \leq 1.26 \frac{M}{\ln M} \quad \text{for } M \geq 17.$$

The following fact is based on a construction, “Freivalds' fingerprinting method”, due to Freivalds [10].

Property 1 *The set F_M of fingerprints is a $(1/c)$ - $U(\pi(M); 2^k, M)$ hash family.*

Proof (sketch). For any pair w, w' of distinct words from $\{0, 1\}^k$ the number $N(w, w') = |\{f_i \in F_M : f_i(w) = f_i(w')\}|$ is bounded from above by k . Thus, if we pick a prime p_i (uniformly at random) from \mathbb{Y} then

$$\Pr[f_i(w) = f_i(w')] \leq \frac{k}{\pi(M)} \leq \frac{k \ln M}{M}.$$

Picking $M = ck \ln k$ for a constant c gives $\Pr[f_i(w) = f_i(w')] \leq \frac{1}{c} + o(1)$. \square

Theorem 4 and Property 1 provide the following statement.

Theorem 6 *Let $c > 1$ be a positive integer and let $M = ck \ln k$. Let $q \in \{M, \dots, 2M\}$ be a prime. Then, for arbitrary $\delta > 0$, family $G = F_M \circ H_{\delta, q}$ is a Δ -R $(2^k; s)$ quantum hash generator, where*

$$s \leq \log ck + \log \log k + \log \log q + 2 \log 1/\delta + 3$$

and

$$\Delta \leq \frac{1}{c} + \delta.$$

Proof. From Theorem 4 we have that

$$s \leq \log \pi(M) + \log \log q + 2 \log 1/\delta + 3.$$

From the choice of c above we have that $M = ck \ln k$. Thus

$$s \leq \log ck + \log \log k + \log \log q + 2 \log 1/\delta + 3.$$

\square

Remark 2 *Note that from Theorem 1 we have*

$$s \geq \log k + \log \log q - \log \log \left(1 + \sqrt{2/(1 - \delta)}\right) - 1.$$

This lower bound shows that the quantum hash function ψ_{F_M} is good enough in the sense of the number of qubits used for the construction.

5.3 Quantum hashing and error-correcting codes

Let q be a prime power and let \mathbb{F}_q be a field. An (n, k, d) error-correcting code is called *linear*, if $\Sigma = \mathbb{F}_q$, and $\mathcal{C} = \{C(w) : w \in \mathbb{F}_q^k\}$ is a subspace of $(\mathbb{F}_q)^n$. We will denote such linear code by an $[n, k, d]_q$ code.

Theorem 7 *Let \mathcal{C} be an $[n, k, d]_q$ code. Then for arbitrary $\delta \in (0, 1)$ there exists a Δ -R $(q^k; s)$ quantum hash generator G , where $\Delta = (1 - d/n) + \delta$ and $s \leq \log n + \log \log q + 2 \log 1/\delta + 4$.*

Proof. The following fact was observed in [5, 13]. Having an $[n, k, d]_q$ code \mathcal{C} , we can explicitly construct a $(1 - d/n)$ -U $(n; q^k; q)$ hash family $F_{\mathcal{C}}$.

By Theorem 4 a composition $G = F_{\mathcal{C}} \circ H_{\delta, q}$ is an Δ -R $(q^k; s)$ quantum hash generator, where $\Delta = (1 - d/n) + \delta$ and $s \leq \log n + \log \log q + 2 \log 1/\delta + 4$. \square

5.3.1 Quantum hash function via Reed-Solomon code

As an example we present construction of quantum hash function, using Reed-Solomon codes.

Let q be a prime power, let $k \leq n \leq q$, let \mathbb{F}_q be a finite field. A *Reed-Solomon* code (for short RS-code) is a linear code

$$C_{RS} : (\mathbb{F}_q)^k \rightarrow (\mathbb{F}_q)^n$$

having parameters $[n, k, n - (k - 1)]_q$. RS-code defined as follows. Each word $w \in (\mathbb{F}_q)^k$, $w = w_0 w_1 \dots w_{k-1}$ associated with the polynomial

$$P_w(x) = \sum_{i=0}^{k-1} w_i x^i.$$

Pick n distinct elements (evaluation points) $A = \{a_1, \dots, a_n\}$ of \mathbb{F}_q . A common special case is $n = q - 1$ with the set of evaluating points being $A = \mathbb{F}_q \setminus \{0\}$. To encode word w we evaluate $P_w(x)$ at all n elements $a \in A$

$$C_{RS}(w) = (P_w(a_1) \dots P_w(a_n)).$$

Using Reed-Solomon codes, we obtain the following construction of quantum hash generator.

Theorem 8 *Let q be a prime power and let $1 \leq k \leq n \leq q$. Then for arbitrary $\delta \in (0, 1)$ there is a Δ -R $(q^k; s)$ quantum hash generator G_{RS} , where $\Delta \leq \frac{k-1}{n} + \delta$ and $s \leq \log(q \log q) + 2 \log 1/\delta + 4$.*

Proof. Reed-Solomon code C_{RS} is $[n, k, n - (k - 1)]_q$ code, where $k \leq n \leq q$. Then according to Theorem 7 there is a family G_{RS} , which is an Δ -R $(q^k; s)$ quantum hash generator with stated parameters. \square

In particular, if we select $n \in [ck, c'k]$ for constants $c < c'$, then $\Delta \leq 1/c + \delta$ for $\delta \in (0, 1)$ and in according to Theorem 1 we get that

$$\log(q \log q) - \log \log \left(1 + \sqrt{2/(1 - \Delta)}\right) - \log c'/2 \leq s \leq \log(q \log q) + 2 \log 1/\Delta + 4.$$

Thus, Reed Solomon codes provides good enough parameters for resistance value Δ and for a number s of qubits we need to construct quantum hash function ψ_{RS} .

Explicit constructions of G_{RS} and $\psi_{G_{RS}}$. Define $(k-1)/q$ -U $(q; \mathbb{F}_q^k; q)$ hash family $F_{RS} = \{f_a : a \in A\}$ based on C_{RS} as follows. For $a \in A$ define $f_a : (\mathbb{F}_q)^k \rightarrow \mathbb{F}_q$ by the rule

$$f_a(w_0 \dots w_{k-1}) = \sum_{i=0}^{k-1} w_i a^i.$$

Let $H_{\delta,q} = \{h_1, \dots, h_T\}$, where $h_j : \mathbb{F}_q \rightarrow \mathbb{F}_q$ and $T = \lceil (2/\delta^2) \ln 2q \rceil$. For $s = \log n + \log T + 1$ composition $G_{RS} = F_{RS} \circ H_{\delta,q}$, defines function

$$\psi_{G_{RS}} : (\mathbb{F}_q)^k \rightarrow (\mathcal{H}^2)^{\otimes s}$$

for a word $w \in (\mathbb{F}_q)^k$ by the rule.

$$\psi_{G_{RS}}(w) = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i\rangle \otimes \left(\frac{1}{\sqrt{T}} \sum_{j=1}^T |j\rangle \left(\cos \frac{2\pi h_j(f_{a_i}(w))}{q} |0\rangle + \sin \frac{2\pi h_j(f_{a_i}(w))}{q} |1\rangle \right) \right).$$

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